Triangulability of compact surfaces

Sidi Mohammed Boughalem (shinokiz@gmail.com)

16. January 2019

In this talk, we will prove that every compact surface (2-manifolds) can be triangulated. This result allows the classification of all compact surfaces. We will do so through topological and combinatorial methods, by introducing the concept of graphs. This result, as we will see, reposes firmly on the *Jordan-Schönflies* theorem, which is a generalization of the well-known *Jordan Curve Theorem*.

1 Preliminaries

Definition 1.1 (Graphs)**.** *A graph G is a triple (V, E, ϕ) where V is a set of vertices, E is a set of edges, and $\phi : E \longrightarrow [V]^2$ is a function assigning to every edge e a two-element set of vertices $\phi(e) = \{u, v\}$ called the endpoints of the edge. We say that the edge e is incident to the two vertices $\{u, v\} \in \phi(e)$. Two edges e and e' are said to be parallel if $\phi(e) = \phi(e')$. A simple graph is a graph with no parallel edges.*

We denote by $[V]^2$ the set of all subsets $\{u, v\}$ consisting of two distinct elements $u, v \in V$. Note that if G is simple, then its set of edges E is a subset of $[V]^2$. We suppose from now on that V and E are finite sets.

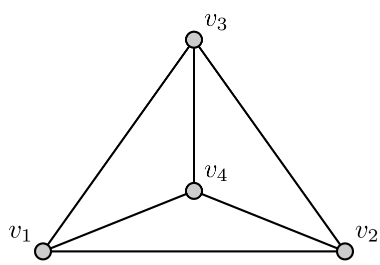

Figure 1: Graph of a triangulation of a sphere

Seminar „Topology vs. Combinatorics“, WS 2018/19, Universität Regensburg

A graph $H = (V_H, E_H, \phi_H)$, is a sub-graph of a graph $G = (V, E, \phi)$ if $V_H \subset V$, $E_H \subset E$ and the restriction of ϕ to E_H is ϕ_H. Given a graph $G = (V, E, \phi)$ and a vertex v we define $G - v$ to be the graph $(V \setminus \{v\}, E', \phi')$, where $E' \subset E$ is obtained by deleting all edges incident to v and $\phi' = \phi_{/E'}$.

Definition 1.2 (Chains, paths, cycles)**.** *A chain is a sequence*

$$\pi = (v_0, e_1, v_1, e_2, ..., v_{n-1}, e_n, v_n), \quad n \geq 1$$

where $v_i \in V$, $e_i \in E$ and $\phi(e_i) = \{v_{i-1}, v_i\}$. (in other words, e_i is an edge with endpoints v_{i-1}, v_i). We say that v_0 and v_n are joined by the chain π. A path is a chain where no vertex v_i occurs twice, that is, if $v_i \neq v_j$ for all $i \neq j$ with $0 \leq i, j \leq n$. A cycle is a chain such that $v_0 = v_n$, $n \geq 2$, and $v_i \neq v_j$ for all $i \neq j$ with $0 \leq i, j \leq n-1$.

Definition 1.3 (Connected graphs)**.** *A graph G is said to be connected if and only if any two distinct vertices of G are joined by a path. It is 2-connected if*

- *G is connected.*
- *G has at least three vertices.*
- *$G - v$ is connected for every vertex v of G.*

Given a graph $G = (E, V, \phi)$, we define an equivalence relation $\backsim$ on its set of edges E as follows: For any two edges $e_1, e_2 \in E$, we say that $e_1 \backsim e_2 \Leftrightarrow$ either $e_1 = e_2$ or there is a cycle containing both e_1 and e_2. Each equivalence class of edges together with all their endpoints is called a *block* of the graph. We agree that every isolated vertex is a block so that every graph is the union of its blocks. An edge e whose equivalence class is reduced to $\{e\}$ is called a *cutedge* . A vertex that belongs to more than one block is called a *cutvertex.*

Exercise 1.4. *show that any two distinct blocks have at most one vertex in common and that such a vertex is a cutvertex.*

The following proposition helps for a better understanding of the definitions above and is not very hard to prove :

Proposition 1.5. *Let G be a connected graph with at least three vertices. Then the following properties are equivalent:*

(i) G is 2-connected.

(ii) Any two vertices of G belong to a common cycle.

(iii) Any two edges of G belong to a common cycle.

(iv) G has no cutvertices.

(v) G has a single block.

Proposition 1.6. *If G is a 2-connected graph, then for any 2-connected sub-graph H of G, the graph G can be build up starting from H by forming a sequence of 2-connected graphs $G_0 = H$, $G_1, ..., G_m = G$, such that G_{i+1} is obtained from G_i by adding a path in G having only its endpoints in G_i, for $i = 0, ..., m-1$.*
In particular, H can be any cycle of length at least three.

Proof. Let $G = (V, E, \phi)$, $H = (V_H, E_H, \phi_H)$ be as above. We proceed by induction on the number of edges in G: If $G = H$ there is nothing to prove. Suppose $H \neq G$, since G is connected, there must be some edge, say $e = \{u, v\} \in E \setminus E_H$, with $u \in V_H$ and $v \in V$. Since G is 2-connected, $G - u$ is connected. Consider a shortest path α in $G - u$ from v to some vertex in V_H. Because this is a shortest path to H, all its edges must be outside E_H and thus α is a path whose endpoints belong to V_H and whose edges are all outside E_H, if we add this path to H we obtain a new 2-connected graph $H_0 = (V_{H_0}, E_{H_0}, \phi_{H_0})$.
Now, $|E - E_{H_0}| < |E - E_H|$, thus by applying the induction hypothesis to H_0 we get the result. □

A *simple polygonal arc* in the plane is a simple continuous curve which is the union of a finite number of straight line segments. A *segment* of a simple closed curve $f : [0, 1] \longrightarrow \mathbb{R}^2$, is either the image $f([a, b])$ or the image $f([0, a] \cup [b, 1])$ for some a, b with $0 \leq a < b \leq 1$.

Definition 1.7 (Topological embedding)**.** *A graph G can be embedded in a topological space X, if the vertices of G can be represented by distinct points of X and every edge e of G can be represented by a simple arc which joins its two endpoints in such a way that any two edges have at most an endpoint in common.*
A planar graph is a graph that can be embedded in $\mathbb{R}^2$, and a plane graph is the image in $\mathbb{R}^2$ of a graph under an embedding.

Given a plane graph G, let $|G|$ be its topological realization, i.e the subset of $\mathbb{R}^2$ consisting of the union of all the vertices and edges of G. it is a compact subset of $\mathbb{R}^2$ and its complement $\mathbb{R}^2 \setminus |G|$, is an open subset of $\mathbb{R}^2$. The arcwise connected components of $\mathbb{R}^2 \setminus |G|$ are called the *faces* of G.
An isomorphism $f : G_1 \longrightarrow G_2$, between two graphs G_1 and G_2 is pair of bijections (f^v, f^e), with $f^v : V_1 \longrightarrow V_2$ and $f^e : E_1 \longrightarrow E_2$, such that for every edge $e \in E_1$:

$$\phi_1(e) = \{a, b\} \Rightarrow \phi_2(f^e(e)) = \{f^v(u), f^v(v)\}$$

Lemma 1.8. *If Ω is any open, arcwise connected subset of $\mathbb{R}^2$, then any two distinct points in Ω are joined by a simple polygonal path.*

Proof. Let $x, \in \Omega$ and let A be a simple arc joining x and in Ω. For each point $z \in \Omega$ there is an $\epsilon > 0$ such that the open disc $D(z, \epsilon) \subset \omega$. By continuity A is compact in Ω hence it has a finite open cover

$$\{D(z_i, \epsilon), z_i \in A\}_{1 \leq i \leq k}$$

Thus, we can now construct a simple polygonal arc in $D_{z_1} \cup ... \cup D_{z_k}$ connecting x and y. □

Note that the above lemma holds for $\Omega = \mathbb{R}^2$. One sees that if G is a planar graph, then G can be drawn (embedded) in the plane so that all edges are simple polygonal arcs, (but we won't need this result).
For the proof of our main theorem, we need a version of Proposition 1.6 for planar graphs. Such a proposition is easily obtained using Lemma 1.8 in the following way : we proceeds by induction as in the proof of Proposition 1.6 but instead of curvy paths added to G_i, we use paths of simple polygonal arcs using Lemma 1.8 and similarly, curvy edges in the cycle H are replaced by paths of simple polygonal arcs.
We obtain thus the following :

Proposition 1.9. *If G is a 2-connected planar graph, then for any 2-connected planar sub-graph H of G, the graph G can drawn in the plane starting from H by forming a sequence of 2-connected plane graphs $G_0, G_1, ..., G_m$ such that G_0 is a planar embedding of H, G_m is a planar embedding of G, and G_{i+1} is obtained from G_i by adding a path consisting of simple polygonal arcs having only their endpoints in G_i, for $i = 0, ..., m-1$.*
In particular, H can be any cycle of length at least three and thus, there is a also a drawing of G in the plane as above where G_0 is a drawing of H with simple polygonal arcs.

2 Jordan Curve theorem and Jordan-Schönflies theorem

The crucial ingredient in the proof that a compact surface can be triangulated is a strong form of the *Jordan curve* theorem known as the *Jordan-Schönflies* theorem.

Theorem 2.1 (Jordan-Schönflies). *If $f : C \longrightarrow C'$ is a homeomorphism between two simple closed curves C and C' in the plane, then f can be extended into a homeomorphism of the whole plane.*

The *Jordan-Schönflies* theorem can be proved in an elementary (but tedious) way in [1] (Chap. II, section 2.2), or using tools from algebraic topology (homology groups). Such a proof can be found in [2] (Chap. IV, Theorem 19.11).
We will need two more technical results :

Theorem 2.2. *If G, G' be 2-connected plane graphs such that g is a homeomorphism and a graph-isomorphism of G onto G'. Then, g can be extended to a homeomorphism of the whole plane.*

Proof. The proof uses *Jordan-Schönflies* theorem and an induction on the number of edges in G.
If G has one cycle, then Theorem 2.2 reduces to The *Jordan-Schönflies* Theorem. If not, then by Proposition 1.9 G has a path P and a 2-connected sub-graph G_1 such that G is obtained from G_1 by adding P to $C \cup \mathring{C}$ where C bounds a face of G_1. As G_1 has fewer cycles than G, we apply the induction hypothesis to G_1 first, and then to the two cycles of $C \cup P$ that contain P. □

Definition 2.3 (Bad, Very bad segments). *Let $\gamma, \gamma_2, \gamma_3 : [0,1] \longrightarrow \mathbb{R}^2$ be three closed simple continuous curves and assume that $\gamma_3([0,1]) \subset \gamma_2([0,1])$.*
We define a bad segment of γ to be a segment segment P joining two points, $p, q \in \gamma_2([0,1])$ with all the other points in $\gamma_2([0,1])$.
We define a very bad segment as a bad segment that intersects $\gamma_3([0,1])$.

Lemma 2.4. *There are only finitely many very bad segments.*

Proof. Since the image of γ is compact and since $\gamma_3([0,1]) \subset \gamma_2([0,1])$, there is some $\epsilon > 0$ so that $\gamma_3([0,1])$ is covered by a finite number of open discs of center in $\gamma_3([0,1])$ and all inside $\gamma_2([0,1])$. Suppose that infinitely many bad segments intersect $\gamma_3([0,1])$ and let $P_1, ..., P_n, ...$ be some infinite sequence of such very bad segments.
Each very bad segment corresponds to two distinct points $p_n = \gamma(u_n)$ and $q_n = \gamma(v_n)$in $\gamma([0,1])$ and we can form the infinite sequence $(t_k)_{k\geq 0}$ with $t_{2k-1} = u_k$ and $t_{2k} = v_k$ for all $k \geq 1$.
Since $[0,1]$ is compact, $(t_k)_{k\geq 0}$ has an accumulation point, say t. By continuity of γ, $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ must have some sub-sequences that converge both to $s = \gamma(t)$. As all the q_n are in $\gamma_2([0,1])$, one has $s \in \gamma_2([0,1])$ and γ intersects γ_2 at s.
Since γ is a continuous, for every $\eta > 0$ there is some ϵ' so that $\gamma(u) \in B(s, \eta)$ for all u with $|u - t| < \epsilon$, which means that some segment P_n is contained in the open disc $B(s, \eta)$. But then, if we choose $\eta < \epsilon$, there is some P_n that do not intersect$\gamma_3([0,1])$, which is a contradiction. Therefore, there are only finitely many very bad segments. □

3 Every compact surface can be triangulated

Definition 3.1. *A surface S is a Hausdorff space which is locally homeomorphic to a disk. In other words, for every point $p \in S$, there exists an open set $U \subset S$ such that $p \in U$ and such that U is homeomorphic to the open disk $D_2 = \{(x, y)/x^2 + y^2 < 1\}$. If S is compact, it is often referred to as a closed surface.*

Example 3.2.

- *The 2-Sphere $\mathbb{S}^2$: The sphere is a closed surface, homeomorphic to $\{(x, y, z) \in \mathbb{R}^3/x^2 + y^2 + z^2 = 1\}$.*
- *The Torus $\mathbb{T}^2$: the torus is a closed surface, that can be obtained in different ways :*
  1. *In $\mathbb{R}^3$, by rotating the circle of equation $(x-2)^2 + z^2 = 1$ around the axis $(0z)$.*
  2. *Through the homeomorphism $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$, where $\mathbb{S}^1$ is the circle.*
  3. *By glueing the two ends of a cylinder in a "natural" way.*
- *The Möbius strip : obtained after glueing the two opposite sides of a rectangle after twisting it.*
- *The Klein bottle $\mathbb{K}^2$: a closed surface that can also be obtained in different ways :*
  1. *By glueing two Möbius strips by their ends.*
  2. *By glueing the two ends of a cylinder in the "opposite" way of the Torus construction.*

Definition 3.3. *Consider a finite set $\mathcal{P}$ of pairwise disjoint convex polygons (together with their interiors) in the plane such that all sides have the same length. Let S be a topological space obtained by gluing polygons $\mathcal{P}$ such that every edge of a polygon $P \in \mathcal{P}$ is identified with precisely one side of another (or the same) polygon. This defines a graph G whose vertices are the corners of the polygons and whose edges are the sides of the polygons. If S is a connected surface (i.e., S is locally homeomorphic to a disc at every vertex v of G) then we say that G is a 2-cell embedding of S. If all the polygons are triangles, then we say that G is a triangulation of S and S is a triangulated surface.*

Theorem 3.4 (Main result)**.** *Every compact (connected) surface S is homeomorphic to a triangulated surface.*

Proof. We prove that S is homeomorphic to a surface with a 2-cell embedding, it is sufficient since the interior of any convex polygon can be triangulated.
We will proceed in three major steps :

- **Setup .**

  For each point $p \in S$, we choose a disc neighbourhood which we think of as an actual disc in the plane, disjoint from all others. Mainly, let $D(p)$ be an open disc in the plane which is homeomorphic to an open subset U_p, with $p \in U_p$ via a homeomorphism, $\theta_p : D(p) \longrightarrow U_p$. Inside $D(p)$ We draw two quadrangles $Q_1(p)$ and $Q_2(p)$ such that $p \in \theta_p(\overset{\circ}{Q_1}(p)) \subset \theta_p(\overset{\circ}{Q_2}(p))$. Since S is assumed to be compact, there is a finite number of points $p_1, ..., p_n$, such that

$$S \subset \bigcup_{i=1}^{n} \theta_{p_i}(\overset{\circ}{Q_1}(p_i))$$

Since $D(p_1), ..., D(p_n)$ are subsets of the plane, we may assume that they are pairwise disjoint as we wanted.
In what follows we are going to keep $D(p_1), ..., D(p_n)$ fixed in the plane but we shall modify the homeomorphisms θ_{p_i} and the corresponding sets $U_{p_i} = \theta_{p_i}(D(p_i))$ on S and consider new quadrangles $Q_1(p_i)$. More precisely, we shall show that $Q_1(p_1), ..., Q_1(p_n)$ can be chosen such that they form a 2-cell embedding of S.

- **The painful step**.
The main difficulty will be arranging things so that the $Q_1(p_i)$'s have only a finite number of points of intersection on S. The idea then is to build up starting with $Q_1(p_1)$ and show how to choose $Q_1(p_2)$ to ensure that $\theta_{p_1}(Q_1(p_1))$ and $\theta_{p_2}(Q_1(p_2))$ only have a finite number of points of intersection inside S.
Suppose, by induction on k, that $Q_1(p_1), ..., Q_1(p_{k-1})$ have been chosen so that any two of $\theta_{p_1}(Q_1(p_1)), ..., \theta_{p_{k-1}}(Q_1(p_{k-1})$ have only a finite number of points in common in S.
We now focus on the "outer square" for p_k, $Q_2(p_k)$. Choose $Q_3(p_k)$ to be a square between $Q_1(p_k)$ and $Q_2(p_k)$. We are going to use $Q_3(p_k)$ to find a "new" $Q_1(p_k)$ which will complete the inductive step. The key notion is the concept of a *bad segment* introduced in Definition 2.3. a *bad segment* is a segment P of some $Q_1(p_j)$ $(1 \leq j \leq k-1)$ such that $\theta_{p_j}(P)$ joins two points of $\theta_{p_k}(Q_2(p_k))$ and has all other points in $\theta_{p_k}(\overset{\circ}{Q_2}(p_k))$. a bad segment inside $Q_2(p_k)$ is *very bad* if $\theta_{p_j}(P)$ also intersects $\theta_{p_k}(Q_3(p_k))$.

  - **CLAIM$_1$:** There may be infinitely many bad segments in $Q_2(p_k)$ but only finitely many very bad ones.

    The reason why there may be infinitely many bad segments is that segments are continuous simple curves and such curves can wiggle infinitely often while intersecting some other continuous simple curve infinitely many times. As $Q_3(p_k) \subset \overset{\circ}{Q_2}(p_k)$, Lemma 2.4 implies that there are only finitely many very bad segments.

Now, since there are only finitely many very bad segments, their union together with $Q_2(p_k)$ form a 2-connected plane graph that we will call G. Using Proposition 1.9, we can redraw G inside $Q_2(p_k)$ such that we get a plane graph G' which is homeomorphic and graph-isomorphic to G and such that all edges of

G' are simple polygonal arcs. Now, we apply Theorem 2.2 to extend the isomorphism from G to G' to a homeomorphism of the interior of $Q_2(p_k)$ and fixing $Q_2(p_k)$. Let $Q'_1(p_k)$ and $Q'_3(p_k)$ be the images of $Q_1(p_k)$ and $Q_3(p_k)$ under this homeomorphism. Note that Q'_1 and Q'_3 are simple closed curves such that $p_k \in \theta_{p_k}(\overset{\circ}{Q'_1}) \subset \theta_{p_k}(\overset{\circ}{Q'_3})$.

- **CLAIM$_2$:** We can now choose a simple closed polygonal curve Q''_3 in $\overset{\circ}{Q_2}(p_k)$ such that $Q'_1 \subset \overset{\circ}{Q''_3}$ and such that Q"$_3$ intersects only the very bad ones. (and is disjoint from any bad segment inside $Q_2(p_k)$).

  Indeed, for every point $q \in Q'_3$, let $R(q)$ be a square with q as midpoint such that $R(q)$ does not intersect either Q'_1 nor any bad segment (only the intersections with the very bad segments are allowed). We consider a (minimal) finite covering of Q'_3 by such squares. The union of those squares is a 2-connected plane graph whose outer cycle can play the role of Q"$_3$.

The main thing we've achieved now is that the very bad segments are now simple polygonal arcs. Now, $G' \cup Q''_3$ is a 2-connected plane graph (either 2-connected or consists of two blocks). If we use Proposition 1.9 we can redraw it so that Q"$_3$ is in fact a quadrangle having Q'_1 in its interior and then use Theorem 2.2 once more to extend this isomorphism to the plane. If we let Q"$_3$ be our "new" choice of $Q_1(p_k)$, then any two of $\theta_{p_1}(Q_1(p_1), ..., \theta_{p_k}(Q_1(p_k)$ have only finite intersections, proving thus the induction assumption.

– **Final step .**
The union of the $\theta_{p_i}(Q_1(p_i))$'s now gives a nice graph Γ on the surface S and the *Jordan-Schönflies* Theorem gives the desired homeomorphism to a 2-cell surface. More precisely, each $Q_2(p_k)$ contains now only finitely many very bad segments, which are all simple polygonal arcs forming a 2-connected plane graph. Each region of the complement of Γ in S is bounded by a cycle C which lives inside some $Q_2(p_k)$. Now we draw a convex polygon C' of side 1 such that the corners of C' correspond to the vertices of C. After appropriate identification of the sides of the polygons C' corresponding to the faces of Γ in S we get a surface S' with a 2-cell embedded graph Γ' which is isomorphic to Γ . This isomorphism of Γ to Γ' may be extended to a homeomorphism f of the embedding of Γ on S onto the embedding of Γ' on S'.
In particular, the restriction of f to the above cycle C is a homeomorphism of C onto C'. By the *Jordan-Schönflies* Theorem, f can be extended to a homeomorphism of the interiors of the C to the interior f C'. This defines a homeomorphism of S onto S' and thus proves the theorem.

$\square$

References

[1] B. Mohar, C. Thomassen. B. Mohar,C. Thomassen. *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, 2001.

[2] Glen E. Bredon. *Topology and Geometry*, GTM No. 139. Springer Verlag, first edition, 1993.